Taylor Series Duality

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7th IEEE Workshop on
Signal Propagation on Interconnects
Hotel Garden, Siena, Italy
May 11-14, 2003

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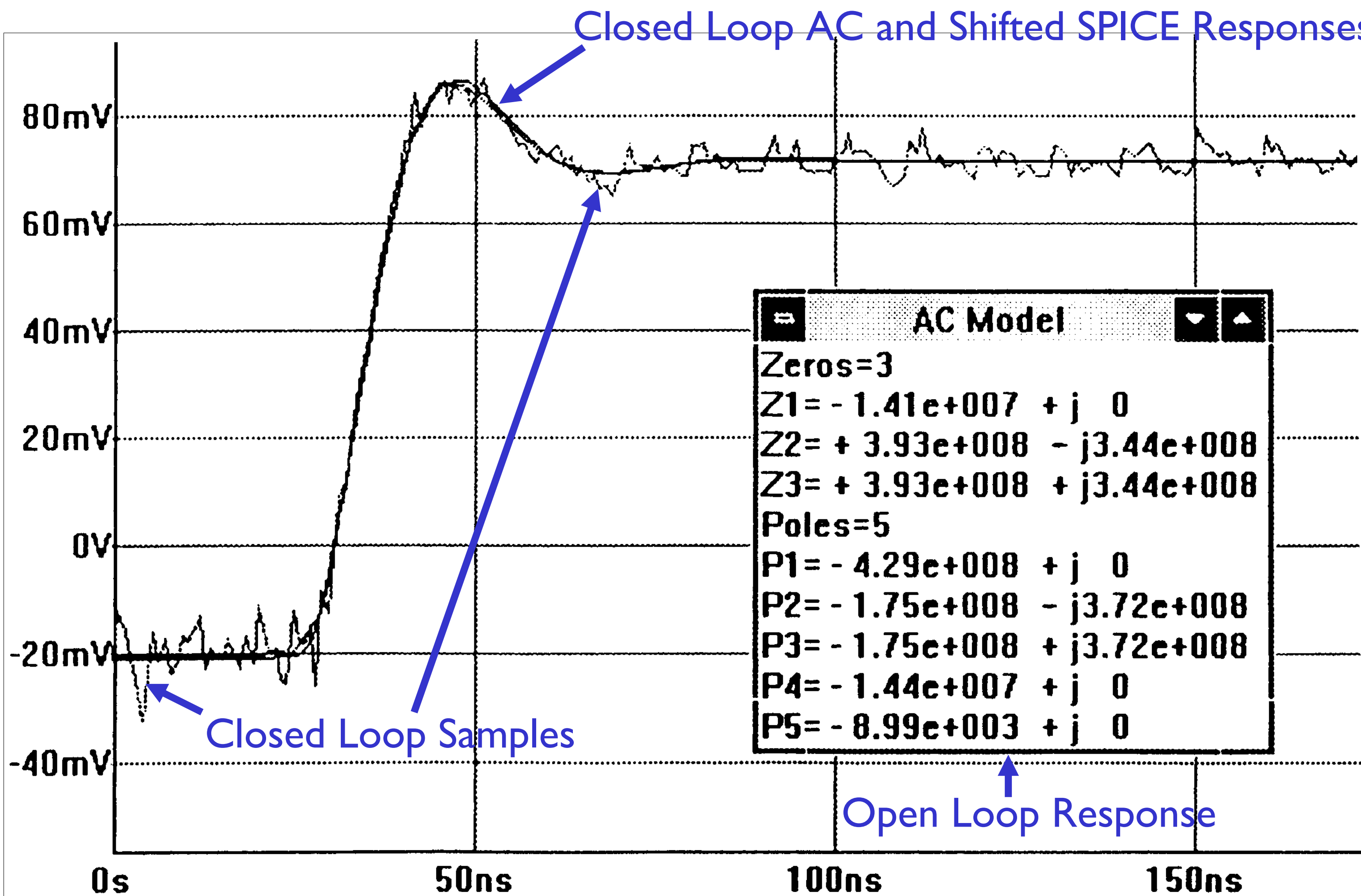
Outline

- Application examples
  - Domain changes and constrained optimization
- Tutorial overview
  - (Known and not so well known relationships)
  - Dual/Transpose symmetry
  - Exponential/Logarithmic relations
- Taylor Series Duality
- Tests

Operational Amplifier Model



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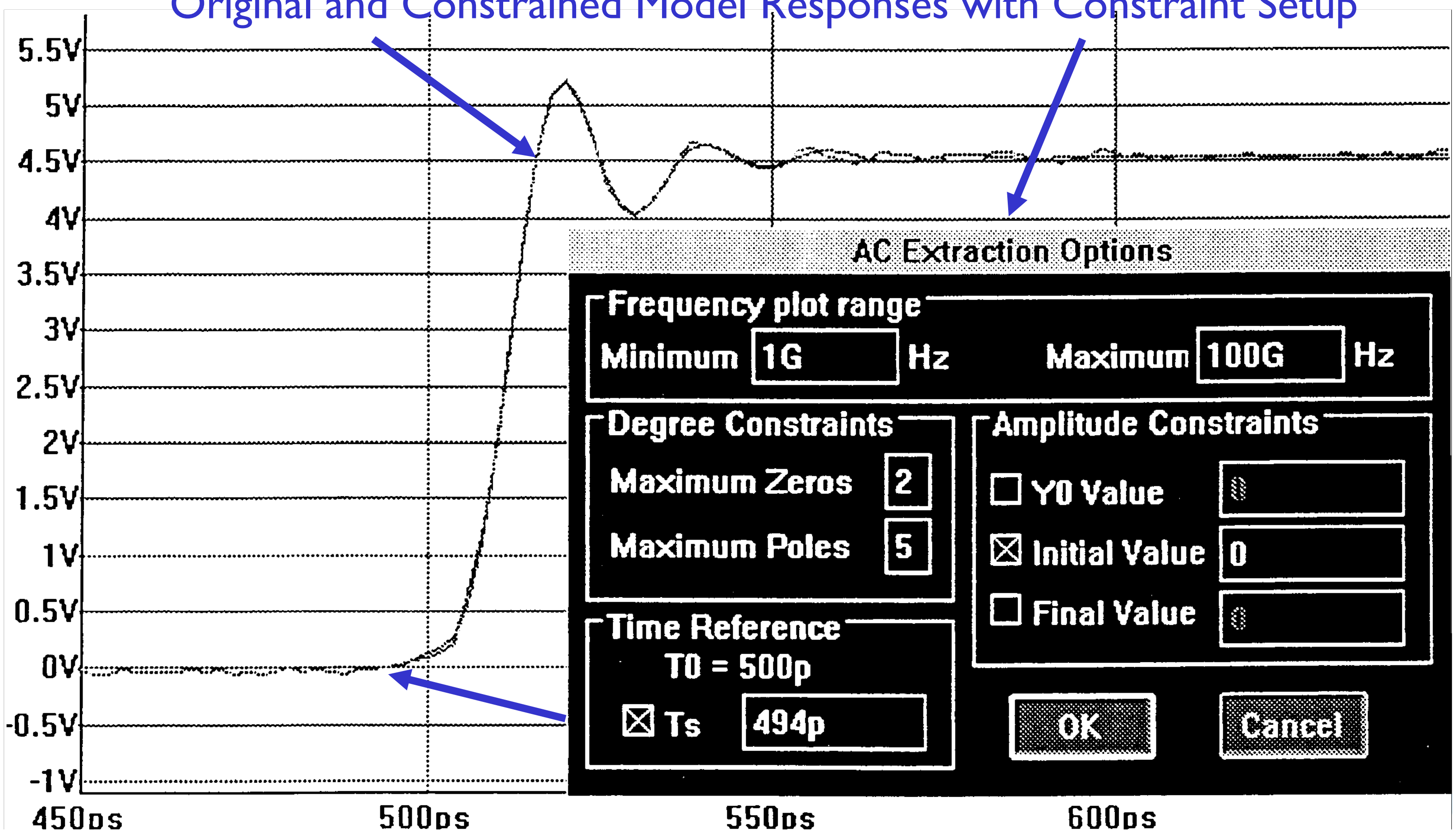
40 GHz Oscilloscope Characterization

Original and Constrained Model Responses with Constraint Setup



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Equivalent Equations (1)-(4)

Laplace Transform	$X(s) = \dfrac{a_{n-1}s^{n-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0},$
Differential Equation	$x^n(t) + b_{n-1}x^{n-1}(t) + \cdots + b_0 x(t) = 0$ initial conditions, $x(0), \cdots, x^{n-1}(0),$
Difference Equation	$x_n(t) + d_{n-1}x_{n-1}(t) + \cdots + d_0 x_0(t) = 0$ initial conditions, $x_0(0), \cdots, x_{n-1}(0),$
Z Transform	$Z(z) = \dfrac{z(c_{n-1}z^{n-1} + \cdots + c_0)}{z^n + d_{n-1}z^{n-1} + \cdots + d_0}.$

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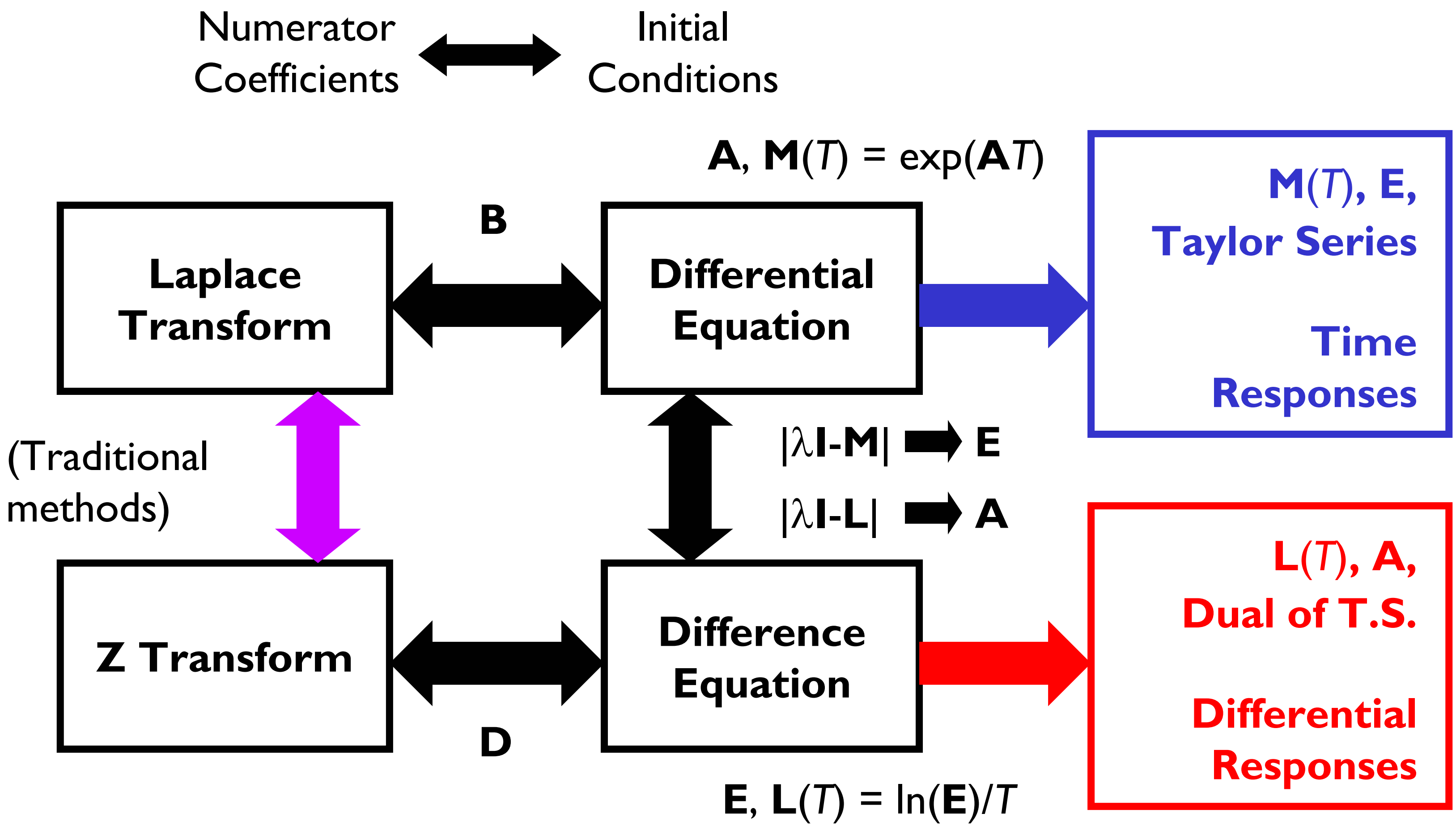
Conversions and Responses (5)-(26)



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Transpose Symmetry (State Transition/Logarithmic Matrices)

Difference (Time) $\mathbf{x}(t)$, $\mathbf{M}(T)$

Differential $\mathbf{z}(t)$, $\mathbf{L}(T)$

$$\begin{matrix} x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\ x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\ x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u \end{matrix}$$

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Transpose Symmetry (Difference/Differential Equation)

Difference (Time) E →

Differential

A ↓

$$
\begin{array}{cccc|ccc}
x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\
x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\
\hline
x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u
\end{array}
$$

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Transpose Symmetry (Taylor Series/Dual Responses)

Taylor Series →

Difference (Time)

E →

Differential

Dual of T.S. ↓

A ↓

$$
\begin{matrix}
x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\
x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\
x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u
\end{matrix}
$$

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Taylor Series and Dual Derivations

- Taylor Series from State Transition Matrix
  - $\mathbf{x}(t+T) = \mathbf{M}(T)\mathbf{x}(t) = (\mathbf{I} + \mathbf{A}T + \mathbf{A}^2T^2/2!\ \ldots)\mathbf{x}(t)$
  - $\mathbf{A}^k\mathbf{x}(t)$ is $\boldsymbol{k}$-th derivative of $\mathbf{x}(t)$
- Dual of T.S. from Natural Logarithm Matrix
  - $d\mathbf{z}(t)/dt = \mathbf{L}(T)\mathbf{z}(t) = -\ [(\mathbf{I} - \mathbf{E}) + (\mathbf{I} - \mathbf{E})^2/2 + (\mathbf{I} - \mathbf{E})^3/3 + \ldots\ ]\mathbf{z}(t)/T$
  - $\mathbf{E}^k\mathbf{z}(t)$ is the $\boldsymbol{k}$-th shifted sample of $\mathbf{z}(t)$
  - Collect the terms for each power of **E**

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Logarithmic Expansion: Binomial Series & Pascal Triangle Reduction

$q,i \downarrow$ / $k \rightarrow$	factor $\downarrow$	0	1	2	3	4	5
0	–	–					
1	$\binom{q}{k}/1$	1/1	1/1				
2	$\binom{q}{k}/2$	1/2	2/2	1/2			
3	$\binom{q}{k}/3$	1/3	3/3	3/3	1/3		
4	$\binom{q}{k}/4$	1/4	4/4	6/4	4/4	1/4	
5	$\binom{q}{k}/5$	1/5	5/5	10/5	10/5	5/5	1/5

$\binom{q}{k}/i$

$q,i \downarrow$ / $k \rightarrow$	0	1	2	3	4	5
factor $\rightarrow$	$\sum_{i=1}^{q} 1/i$	$\binom{q}{1}/1$	$\binom{q}{2}/2$	$\binom{q}{3}/3$	$\binom{q}{4}/4$	$\binom{q}{5}/5$
0	–					
1	1/1	1/1				
2	3/2	2/1	1/2			
3	11/6	3/1	3/2	1/3		
4	25/12	4/1	6/2	4/3	1/4	
5	137/60	5/1	10/2	10/3	5/4	1/5

$\binom{q}{k}/k$

$$\beta_k = \frac{(-1)^{k-1}}{T} \sum_{i=k}^{q} \binom{q}{k} \Big/ i \quad k > 0$$

$$\beta_k = \frac{(-1)^{k-1}}{T} \binom{q}{k} \Big/ k, \quad k > 0$$

Correction: replace *q* with *i*

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Example: sin($10\pi t$), Some Original & Scaled Terms

- Scaling by weighting samples: $y_i = \exp(-\gamma t)x_i$

k	$T\beta_k$	Scaled $T\beta_k$ (41 terms)
0	-3.14586	-3.14586
1	40.	12.8867
9	3.04e+7	1135.93 (maximum value)
20	-6.89e+9	-1.00 (set by scaling)
30	-2.83e+7	-4.93801e-8
40	-0.025	-5.26270e-22

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Example: sin($10\pi t$), Last 1/2 Cycle, 48-th Derivative

- $0.0 <= t <= 1.0$, $T = 0.02$, 51 samples

Function	Exact	Dual T.S. (error bold)
• sin[45]	0.0000**00**	**-1.08745e-5**
• sin[46]	-0.587**785**	-0.587**844**
• sin[47]	-0.951**057**	-0.951**141**
• sin[48]	-0.951**057**	-0.951**134**
• sin[49]	-0.587**785**	-0.587**827**
• sin[50]	0.0000**00**	**1.08745e-5**

– (All other iterative methods are “Exact”)

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Example: sin($10\pi t$), Last 1/2 Cycle, 49-th Derivative

- 0.0 <= *t* <= 1.0, *T* = 0.02, 51 samples

Function	Exact	Dual T.S. (error bold)
cos[45]	-1.000**00**	-1.000**90**
cos[46]	-0.8090**17**	-0.8090**81**
cos[47]	-0.3090**17**	-0.3090**33**
cos[48]	0.3090**17**	0.3090**55**
cos[49]	0.8090**17**	0.8090**94**
cos[50]	1.000**00**	1.000**90**

– (All other iterative methods are "Exact")

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Conclusions

- "Tutorial" on exact transformations
  - Practical modeling applications
  - Common routines for both domains
- Taylor Series & Binomial Series "duality"
  - Accurate with scaling
  - But not as accurate and stable as other iterative methods

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