Taylor Series Duality

Bob Ross
Teraspeed Consulting Group
Portland, Oregon USA
bob@teraspeed.com

Presented at the
7th IEEE Workshop on
Signal Propagation on Interconnects
Hotel Garden, Siena Italy
May 11-14, 2003

Corrections:

Changed "parallel" to "parallelism" in **Notation and Existing Relations**

Equation 29: Replaced q with i in the binary series coefficient expression

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Abstract

This paper introduces a "dual" formulation of the Taylor series when applied to network function evaluation methods. Many applications use network functions given as a Laplace transform ratio of polynomials in s. For such a network function (and its corresponding differential equation), a recursive Taylor series expansion method exists to calculate the time response and derivatives at the next time interval. The main result of this paper is to show a dual recursive formulation for the equivalent Z transform (and its corresponding difference equation) to evaluate the next set of higher order derivatives for all the data points. The dual terms corresponding to the Taylor series expansion contain binomial series coefficients divided by the coefficient indices. Experimental validation shows good results using a suggested scaling method for in-place calculations, but these results are not quite as accurate as other equivalent iterative methods.

Introduction

The Taylor series expansion is used in many applications, such as reduced order modeling by the Asymptotic Waveform Evaluation (AWE) method to calculate moments [1]. For a given a network function given as a Laplace transform ratio of polynomials in s (and its corresponding differential equation), a recursive Taylor series method exists [2] to evaluate the time response (and corresponding set of derivatives) at the next time interval. The main result of this paper is expressed in (25)-(26). This result shows a "dual" recursive formulation for the equivalent Z transform (and its corresponding difference equation) to successively evaluate the derivatives of the data point set. The dual terms corresponding to Taylor series coefficients contain binomial series coefficients divided by the coefficient indices. To support these results, the paper extends known relations, some of which are captured in [3], between differential and difference equation domains and the corresponding power series expansions, such as in [4]. A scaling method is presented for practical applications of the dual formulation.

Experimental validation shows good results using a suggested scaling method, but these results are not quite as accurate as other equivalent iterative methods. The topics of this paper are presented in this order: Notation and Existing Relations, Main Result, Derivation Outline, Scaling, Experimental Results, and Conclusion.

Notation and Existing Relations

The following unconventional notation emphasizes the parallelism within the differential and difference equation domains and corresponding transformations. Let the $i\text{-}th$ derivative with respect to time be denoted as $x^i(t) = dx^i(t)/dt^i$, and let the sample data values at time T intervals be denoted as $x_k(t) = x(t+kT)$. Then the following definitions are equivalent for a network function Laplace transform (1), differential equation (2), difference equation (3), and Z transform (4):

$$X(s) = \frac{a_{n-1}s^{n-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0}, \tag{1}$$

$$x^n(t) + b_{n-1}x^{n-1}(t) + \cdots + b_0 x(t) = 0 \tag{2}$$

and initial conditions, $x(0), \cdots, x^{n-1}(0)$,

$$x_n(t) + d_{n-1}x_{n-1}(t) + \cdots + d_0 x_0(t) = 0 \tag{3}$$

and initial conditions, $x_0(0), \cdots, x_{n-1}(0)$,

$$Z(z) = \frac{z(c_{n-1}z^{n-1} + \cdots + c_0)}{z^n + d_{n-1}z^{n-1} + \cdots + d_0}. \tag{4}$$

Previously documented relationships between (1) through (4) along with time response evaluations are presented to support further development. Vectors and matrices are shown by **bold** notation, and the exponent T designates transpose. Equations (5)-(8) relate differential equation initial conditions with the Laplace transform numerator coefficients:

$$\mathbf{x}(t) = [x^0(t), x^1(t), \cdots, x^{n-1}(t)]^T, \tag{5}$$

$$\mathbf{a} = [a_{n-1}, \cdots, a_0]^T, \tag{6}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ b_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \cdots & 1 & 0 \\ b_1 & b_2 & \cdots & b_{n-1} & 1 \end{bmatrix}, \tag{7}$$

$$\mathbf{a} = \mathbf{Bx}(0). \tag{8}$$

In a parallel manner, (9)-(12) relate difference equation initial conditions with Z transform numerator coefficients:

$$\mathbf{z}(t) = [x_0(t), x_1(t), \cdots, x_{n-1}(t)]^T, \tag{9}$$

$$\mathbf{c} = [c_{n-1} \cdots c_0]^T, \tag{10}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ d_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_2 & d_3 & \cdots & 1 & 0 \\ d_1 & d_2 & \cdots & d_{n-1} & 1 \end{bmatrix}, \tag{11}$$

$$\mathbf{c} = \mathbf{D}\mathbf{z}(0). \tag{12}$$

A well-known method [4] to evaluate the transient response of (1) or (2) is expressed using the state transition matrix in (13)-(16):

$$d\mathbf{x}(t)/dt = \mathbf{A}\mathbf{x}(t), \tag{13}$$

where the companion matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{bmatrix}. \tag{14}$$

Starting with the initial conditions of (2), the time response is found recursively from

$$\mathbf{x}(t+T) = \mathbf{M}\mathbf{x}(t), \tag{15}$$

where the matrix exponential function of $\mathbf{A}$ is

$$\mathbf{M} = \exp(\mathbf{A}T). \tag{16}$$

In the proposed computational method that bypasses calculating roots of (1) or (2), $\mathbf{M}$ is evaluated by the power series given later in (27).

The traditional method to convert to/from differential and difference equations is to do partial fraction expansions of the Laplace or Z transforms based on factorization of their denominators. The terms are converted to the other domain by conversion tables or formulas. This is an exact process, but it becomes tedious for general n-th order functions with multiple and complex poles. Other methods are based on the substitution of $s = \ln(z)/T$ (or its inverse) directly into the Laplace (or Z) transform where $\ln(z)$ is the natural logarithm of z. Several approximations can be used including one referred to as a bilinear mapping.

Another approach is based on directly converting the differential equation to the difference equation. The process is derived from the Cayley-Hamilton theorem stating that a matrix satisfies its own characteristic equation. Applying this theorem yields the computational relationship between differential equation and difference equation coefficients:

$$|l\mathbf{I} - \mathbf{M}| = l^n + d_{n-1} l^{n-1} + \cdots + d_0 = 0, \tag{17}$$

where $\mathbf{I}$ is the unit matrix. The initial values for the difference equation are obtained by successively applying (15).

The inverse process can be formulated in a parallel manner. Starting with its initial conditions, the difference equation time response given by (3) is already of the form:

$$\mathbf{z}(t+T) = \mathbf{E}\mathbf{z}(t), \tag{18}$$

where the companion matrix $\mathbf{E}$ is

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & \cdots & -d_{n-1} \end{bmatrix}. \tag{19}$$

Equation (18) is the solution to

$$d\mathbf{z}(t)/dt = \mathbf{L}\mathbf{z}(t), \tag{20}$$

where the matrix exponential and its corresponding natural logarithm function of $\mathbf{E}$ is

$$\mathbf{E} = \exp(\mathbf{L}T) \text{ and } \mathbf{L} = \ln(\mathbf{E})/T. \tag{21}$$

The Cayley-Hamilton theorem yields the conversion from difference equation coefficients to differential equation coefficients:

$$|l\mathbf{I} - \mathbf{L}| = l^n + b_{n-1} l^{n-1} + \cdots + b_0 = 0. \tag{22}$$

The initial conditions for the differential equation are found by successively applying (20).

Main Result

The main result is presented in terms of combined notation for the *i-th* derivative of the *j-th* sample: $x_j{}^i(t) = d^i x(t + jT)/dt^i$. A recursive, in-place time response evaluation based on p+1 terms of a Taylor series expansion (23)-(24) has been reported [2]:

$$x_{j+1}{}^{i}(t) = \sum_{k=0}^{p} x_j{}^{i+k}(t) a_k , \qquad (23)$$

where

$$a_k = T^k / k! . \qquad (24)$$

Although another useful Taylor series formulation uses all available derivatives, the summation form (23)-(24) for each *i-th* derivative has a dual formulation given by (25)-(26) as the main result. This dual form describes the series expansion of difference equations used to calculate the next higher order set of derivatives from the previous q+1 sample points:

$$x_j{}^{i+1}(t) = \sum_{k=0}^{q} x_{j+k}{}^{i}(t) b_k , \qquad (25)$$

where

$$b_0 = -\frac{1}{T} \sum_{i=1}^{q} 1/i ,$$

$$b_k = \frac{(-1)^{k-1}}{T} \binom{q}{k} \Big/ k , \ k > 0. \qquad (26)$$

Each b_k term contains a binomial coefficient:

$$\binom{q}{k} = \frac{q!}{k!\,(q-k)!} = \binom{q}{q-k}.$$

Derivation Outline

Equations (23)-(24) can be derived from the power series expansion for the state transition matrix in (15)-(16):

$$\begin{aligned} \mathbf{x}(t+T) &= \mathbf{M}(T)\mathbf{x}(t) \\ &= [\mathbf{I} + \mathbf{A}T + \mathbf{A}^2T^2/2! + \cdots]\mathbf{x}(t). \end{aligned} \qquad (27)$$

The increasing powers of $\mathbf{A}$ in (27) produce successively higher order derivatives of $\mathbf{x}(t)$. The rows in (27) yield equations (23)-(24) for evaluating the *0-th* through the *(n-1)-th* derivative.

Similarly, (25)-(26) can be derived from the power series expansion shown in (28) for the natural logarithm matrix in (20)-(21):

$$\begin{aligned} d\mathbf{z}(t)/d(t) &= \mathbf{L}(T)\mathbf{z}(t) \\ &= -[(\mathbf{I} - \mathbf{E}) + (\mathbf{I} - \mathbf{E})^2/2 \\ &\quad + (\mathbf{I} - \mathbf{E})^3/3 + \cdots]\mathbf{z}(t)/T. \end{aligned} \qquad (28)$$

The increasing powers of $\mathbf{E}$ in (28) provide successively shifted sample points of $\mathbf{z}(t)$. The rows in (28) yield equations (25)-(26) after expanding the powers of $(\mathbf{I} - \mathbf{E})$ and collecting the factors of the powers $\mathbf{E}$. This process produces the summations containing binomial coefficients:

$$b_k = \frac{(-1)^{k-1}}{T} \sum_{i=k}^{q} \binom{i}{k} \Big/ i , \ k > 0. \qquad (29)$$

The summation in (29) is simplified to the final form given in (26). The equivalence of these two forms can be proven by mathematical induction. Figure 1 shows the indexing convention and binomial coefficient factors in a Pascal's triangle, left-justified format.

$q,i \downarrow / k \rightarrow$	*factor* $\downarrow$	0	1	2	3	4	5
0	–	–					
1	$\binom{q}{k}/1$	1/1	1/1				$\binom{q}{k}/i$
2	$\binom{q}{k}/2$	1/2	2/2	1/2			
3	$\binom{q}{k}/3$	1/3	3/3	3/3	1/3		
4	$\binom{q}{k}/4$	1/4	4/4	6/4	4/4	1/4	
5	$\binom{q}{k}/5$	1/5	5/5	10/5	10/5	5/5	1/5

$q,i \downarrow / k \rightarrow$	0	1	2	3	4	5
factor $\rightarrow$	$\sum_{i=1}^{q} 1/i$	$\binom{q}{1}/1$	$\binom{q}{2}/2$	$\binom{q}{3}/3$	$\binom{q}{4}/4$	$\binom{q}{5}/5$
0	–					
1	1/1	1/1				$\binom{q}{k}/k$
2	3/2	2/1	1/2			
3	11/6	3/1	3/2	1/3		
4	25/12	4/1	6/2	4/3	1/4	
5	137/60	5/1	10/2	10/3	5/4	1/5

Figure 1: Example showing factors of (29) and (26).

Scaling

Power series expansions of logarithmic functions converge when the argument is within a defined region. In practice, convergence can be handled by scaling. The unscaled form of (26) yields large values that are unsuitable for numerical processing. A suggested scaling method is presented to achieve power series convergence. Let a scaled response be defined by

$$y_j^{\,i}(t) = \exp(-gt)\,x_j^{\,i}(t)\,, \qquad (30)$$

then

$$y_j^{\,i+1}(t) = -g\,\exp(-gt)\,x_j^{\,i}(t) + \exp(-gt)\,x_j^{\,i+1}(t)\,,$$

or

$$x_j^{\,i+1}(t) = g x_j^{\,i}(t) + \exp(\,l\,t)\,y_j^{\,i+1}(t)\,. \qquad (31)$$

For calculations involving (25) and (26), the scaling process can be made computationally equivalent to producing modified b'_k terms. Since $b_1 T = q$, a convenient scaling choice is to equate:

$$\exp(gT) = q \text{ and } b'_k = b_k / q^k\,. \qquad (32)$$

With this method, the b'_k terms decrease very rapidly as k increases. An in-place iterative process is created using the first term adjustment shown in (31).

Experimental Results

Sine and cosine functions expressed as second order Laplace and Z transforms were used to test the accuracy of the methods. The time step T was selected as an integer divisor of the period so that the exact, cyclic results were known for any data point and any derivative.

Several iterative methods were carried out to the *50-th* derivative and *50-th* time point with 41 term series expansions. The recursive method described by (25)-(26) and scaling (31)-(32) produced results tending to deviate at the third or fourth decimal digit while other iterative methods produced exact or nearly exact results to six decimal digits. The results were sensitive to scaling and could be tuned to be more accurate with less aggressive scaling to four decimal digit accuracy. The results were also sensitive to the T selection and were more accurate for larger values. More investigation is needed, but the overall assessment is that the new method is not as numerically stable as other methods.

Conclusion

The parallel representation of network function differential and difference domain expressions serves as a basis for deriving a dual formulation of a Taylor series recursive time response evaluation. The dual formulation recursively calculates increasing order derivatives of difference domain samples. In this context, the Taylor series has a dual formulation containing binomial coefficients divided by their indices. An experimental test case shows good results with a suggested scaling method. However, these results are not as accurate as those produced by other iterative methods.

References

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